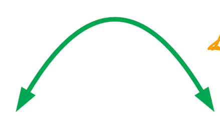
MATH1520 University Mathematics for Applications **Spring 2021**

Chapter 7: Application of Derivatives II

Learning Objectives:
(1) Discuss concavity.
(2) Use the sign of the second derivative to find intervals of concavity.
(3) Locate and examine inflection points.
(4) Apply the second derivative test for relative extrema.
(5) Determine horizontal and vertical asymptotes of a graph.
(6) Discuss and apply a general procedure for sketching graphs.

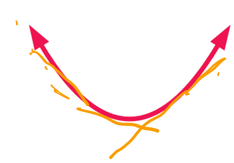
7.1 Concavity and points of inflection

Intuitively: On the $x - y$ plane: when a curve, or part of a curve, has the shape:



(concave) (convex upward)

we say that the shape is concave downward. On the other hand, if it takes the shape



we say that it is concave upward. (convex) (convex downward)

Remark. In some textbooks "concave upward" is called concave up or convex; "concave downward" is called concave down or concave.

Definition 7.1.1. If the function $f(x)$ is differentiable on the interval (a, b), then *the graph* of f is

(i) strictly concave upward on (a,b) if $f'(x)$ is strictly increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) > 0$. on (a,b)

(ii) strictly concave downward on (a,b) if $f'(x)$ is strictly decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) < 0$.

(iii) concave upward on (a,b) if $f'(x)$ is increasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \geq 0$.

(iv) ~~strictly~~ concave downward on (a,b) if $f'(x)$ is ~~strictly~~ decreasing on the interval. In particular, if f is second-differentiable, the condition is equivalent to $f''(x) \leq 0$.

if the graph of f is convex/strictly convex

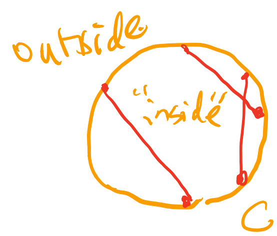
In case (i)/(iii), the function f is said to be *convex/strictly convex*; in case (ii)/(iv), f is said to be *concave/strictly ~~convex~~*. concave.

Remark. 1. In some calculus texts, what we called "strictly convex/concave" above is called "convex/concave". and what we called "convex/concave" is called "weakly convex/concave"

2. General definition of convexity/concavity of continuous curves on a plane via secant lines:

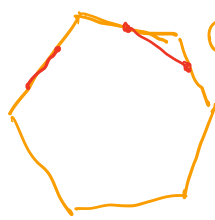
- For continuous closed curves, E.g, a circle C on the plane:

not the graph of a function



is strictly convex because all secant lines to C lies in the "inside" of this closed curve C

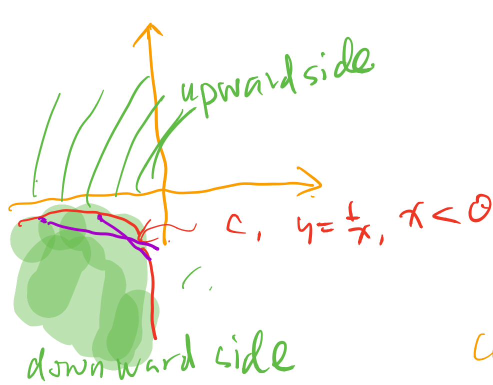
E.g,



C is polygone, "convex" (all secant lines do not intersect the "outside" of C

- When C is the graph of a function f, on the x-y plane:

E.g, $C = \{(x,y) \mid y = \frac{1}{x},\ x<0\}$



C is strictly concave

any secant line of the graph lies in the downward side of C

Concavity, (strict) convexity etc are defined in a similar manner

A test for shapes of graphs:

E.g., $f(x) = -x^2$ $f' = -2x$ strictly decreasing $f'' = -2$

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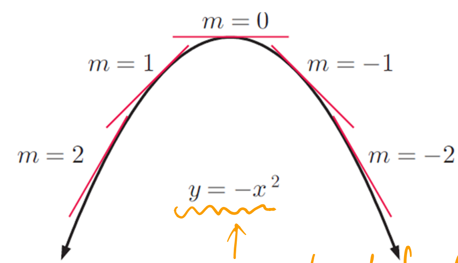


the graph of f described by this equation "strictly concave"

As x increases, $f'(x)$ is $\downarrow$

$f''(x) = -2 < 0$ for strictly concave downward curve.

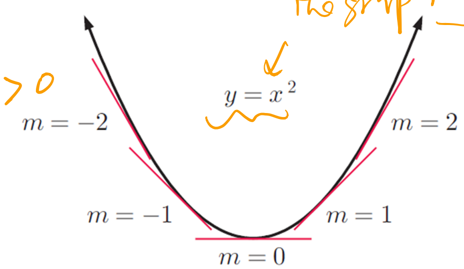
E.g., $f(x) = x^2$.

$f' = x$

$f'' = 2 > 0$

the graph of f is described by this equ.

is strictly convex" (therefore also convex)



As x increases, $f'(x)$ is $\uparrow$

$f''(x) = 2 > 0$ for strictly concave upward curve.

Definition 7.1.2. If $f(x)$ changes strict concavity at some point c in the domain, then the point $(c, f(c))$ on the $x - y$ plane is called an *inflection point* of the graph of f.

$f : (a, b) \to \mathbb{R}$

$c \in (a, b)$

$f(x)$ is strictly convex on (a, c).

but strictly concave on (c, b)

or " concave "

convex "

the the point $(c, f(c))$ on the graph of f is an "inflection point" of this graph

Procedure for Determining Intervals of Concavity & Inflection Points:

Suppose the function $f(x)$ is such that f'' is piecewise continuous.

1. Find all c for which $f''(c) = 0$ or $f''(c)$ does not exist, and divides the domain into several intervals.
2. For each interval,
   - if $f''(x) > 0$, the graph of $f(x)$ is strictly concave upward. (I.e. f is a convex function.)
   - if $f''(x) < 0$, the graph of $f(x)$ is strictly concave downward. (I.e. f is a concave function.)
3. For all c found in step 1,
   - if $f''(x)$ changes sign on two sides of c, then $(c, f(c))$ is an inflection point on the graph of f;
   - otherwise, $(c, f(c))$ is not an inflection point on the graph of f.

Example 7.1.1.

$$f(x) = x^3 + 1 \qquad f' = 3x^2 \qquad f(0) = 1$$

$$f''(x) = 6x = 0 \quad \Rightarrow \quad x = 0.$$

- if $x < 0$, $f''(x) < 0$, $\Rightarrow$ f is strictly concave on $(-\infty, 0)$;
- if $x > 0$, $f''(x) > 0$, $\Rightarrow$ f is strictly convex on $(0, \infty)$.

Since $f''(x)$ changes signs on both sides of $x = 0$, $(0, 1)$ is the unique inflection point on the graph of f.

$(0, f(0))$

Example 7.1.2. Describe the concavity and find all inflection points of the graph of $f(x) = 2x^6 - 5x^4 + 7x - 3$.

$f' = 12x^5 - 20x^3 + 7$

$\frac{d}{dx}$

Solution.

>0 on the interior of the intervals

f'' is well-defined on $\mathbb{R}$

$$f''(x) = 60x^4 - 60x^2 = 60x^2(x^2 - 1) = 60x^2(x-1)(x+1) = 0 \quad \Rightarrow \quad x = 0, \pm 1.$$

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
$f''(x)$	$+$	0	$-$	0	$-$	0	$+$
concavity	up($\smile$)		down($\frown$)		down($\frown$)		up($\smile$)

Two inflection points: $(-1, -13)$, $(1, 1)$.
($(0, -3)$ is not an inflection point!)

not

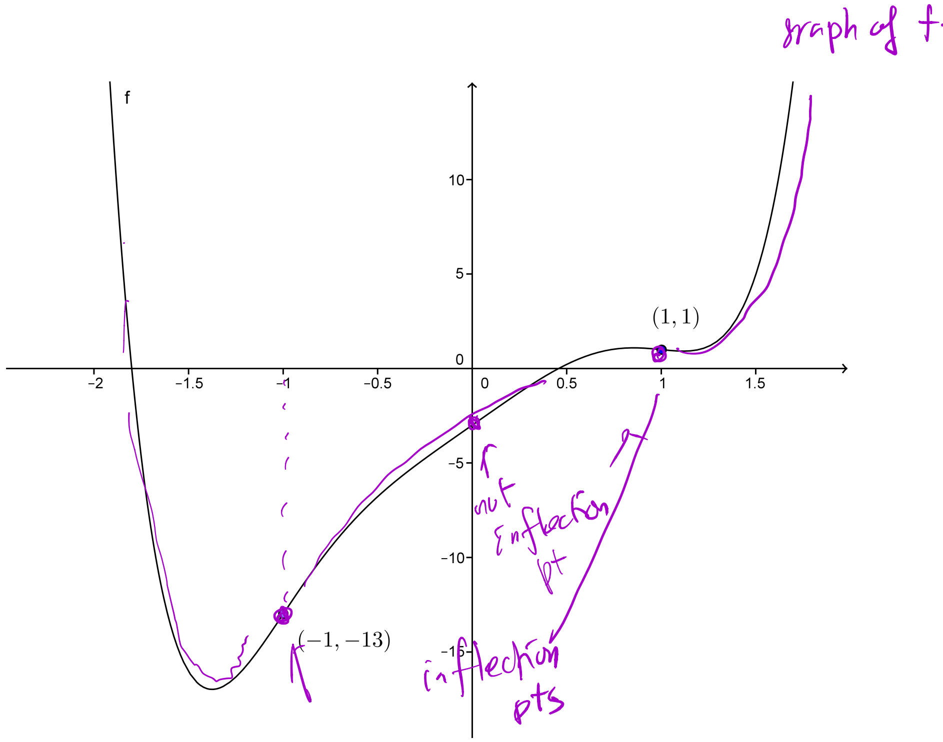
f strictly convex, strictly concave, strictly concave, strictly convex

x

f'': $+$ -1 $-$ 0 $-$ 1 $+$

inflection

points.



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Remark.

- c is a critical point $\iff$ $f'(c) = 0$ or $f'(c)$ does not exist
  c is a critical point $\left\{ \begin{array}{c} \Leftarrow \\ \not\Rightarrow \end{array} \right\}$ f' changes sign at c
- $(c, f(c))$ is an inflection point $\iff$ f'' changes sign at c
  $(c, f(c))$ is an inflection point $\left\{ \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \right\}$ $f''(c) = 0$ or $f''(c)$ does not exist

Theorem 7.1.1 (The Second Derivative Test: Relative Extrema).

Suppose $f'(a) = 0$!

1. *If $f''(a) < 0$, then f has a relative maximum at a.*
2. *If $f''(a) > 0$, then f has a relative minimum at a.*
3. *If $f''(x) = 0$, we have no conclusion.*